Sinyaller ve Sistemler

"Z-Dönüşümü"

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İçindekiler

- Z-Dönüşüm
- Z-Dönüşüm Özellikleri
- Region of Convergence (ROC)
- Block Diagram Manipulation
- Inverse Z-Transform
- Solving Difference Equations
- Transfer Function

Z-DÖNÜŞÜM

- Sürekli zamanlı sinyallerin zaman alanından frekans alanına geçişi Fourier ve Laplace dönüşümleri ile mümkün olmaktadır.
- Laplace, Fourier dönüşümünün genel bir şeklidir.
- Ayrık zamanlı sinyaller için de ayrık zamanlı Fourier dönüşümleri kullanılmaktadır.
- z dönüşüm de ayrık zamanlı Fourier dönüşümünün genel bir şeklidir.
- Ayrık zamanlı f(n) sinyalinin (veya dizinin) z dönüşümü aşağıdaki gibi ifade edilir.

Basic Concepts

- Consider a sequence of values: $\{x_k : k = 0,1,2,... \}$
- These may be samples of a function x(t), sampled at instants t = kT; thus $x_k = x(kT)$.
- The Z transform is simply a polynomial in *z* having the x_k as coefficients:

$$X(z) = Z\{x_k\} = \sum_{k=0}^{\infty} x_k z^{-k}$$

Z-Transform

- Fourier Transform

$$X\left(e^{jw}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn}$$

- z-transform

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

Bilateral - Unilateral

- Two sided or bilateral z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Unilateral z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Z-Dönüşümü

- Ayrık değer ve indisleri birlikte verilir.
- İndisler positif ise üsler negatif; indisler negatif ise üsler pozitif.
- İndisler sıralı alınır. …, -5,-4,-3,-2,-1,0,1,2,3,…
- Olmayan indislerin çarpım katsayıları sıfır alınır. Diğer katsayılar olduğu gibi alınır.

Example of z-transform

n	$n \le -1$	0	1	2	3	4	5	$N > 5$
$x[n]$	0	2	4	6	4	2	1	0

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

Unit Step Signal - continued

A little bit of math ...

$$1 + a + a^2 + \ldots + a^n = \frac{(1-a)(1+a+a^2+\ldots+a^n)}{1-a}$$

$$= \frac{1-a^{n+1}}{1-a}$$

$n \rightarrow \infty$, assuming $|a| < 1$,

$$1 + a + a^2 + \ldots = \lim_{n\to\infty} \frac{(1-a)(1+a+a^2+\ldots+a^n)}{1-a}$$

$$= \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$$

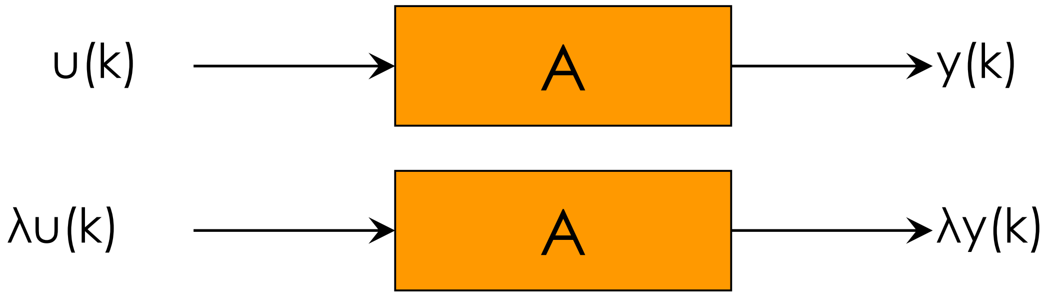
$$= \frac{1}{1-a}$$

$$U_{step}(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots = \frac{1}{1-z^{-1}}$$

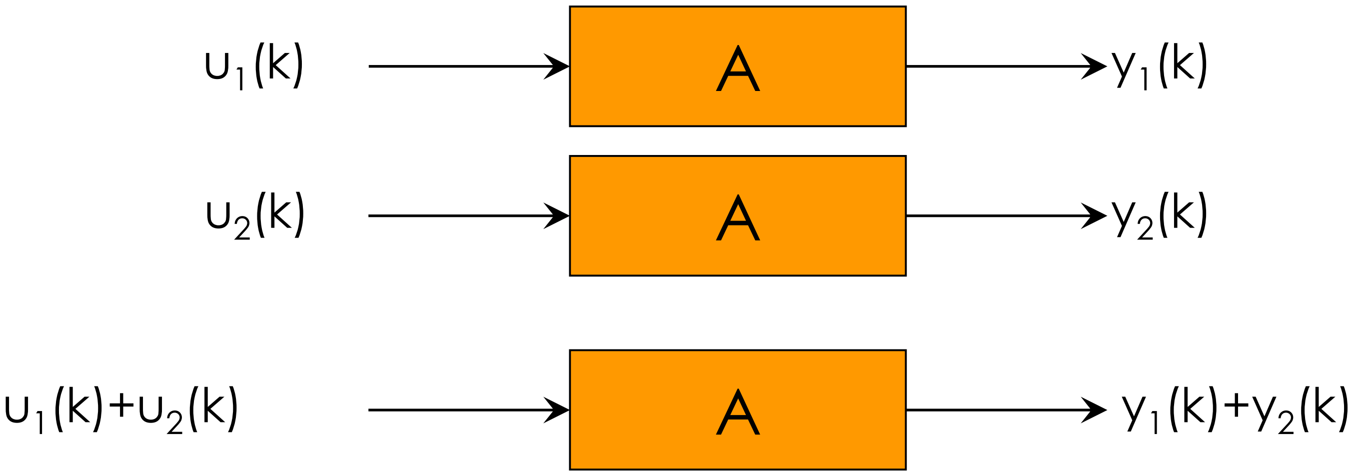
Z-Dönüşüm Özellikleri

What does "Linear" mean exactly?

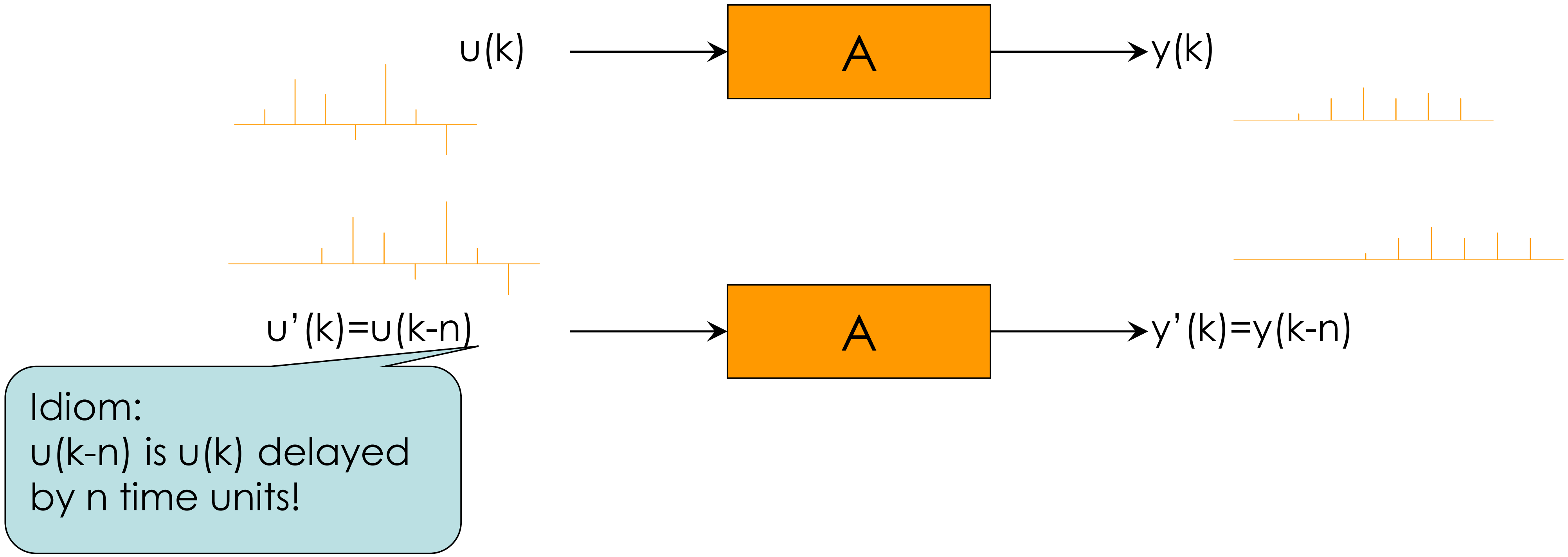
- Scaling



- Superposition



Time Invariance



Properties of the Z-Transform

- **Linearity:** $ax_1[n] + bx_2[n] \quad \Leftrightarrow \quad aX_1[z] + bX_2[z]$

  Proof:

$$\mathcal{Z}\{ax_1[n] + bx_2[n]\} = \sum_{n=-\infty}^{\infty}(ax_1[n] + bx_2[n])z^{-n} = \sum_{n=-\infty}^{\infty} ax_1[n]z^{-n} + \sum_{n=-\infty}^{\infty} bx_2[n]z^{-n} = X_1[z] + X_2[z]$$

- **Time-shift:** $x[n - n_0] \quad \Leftrightarrow \quad z^{-n_0} X[z]$

  Proof:

$$\mathcal{Z}\{x[n - n_0]\} = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x[m]z^{-m} z^{-n_0} = z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-n_0} X[z]$$

  What was the analog for CT signals and the Laplace transform?

- **Multiplication by *n*:** $nx[n] \quad \Leftrightarrow \quad -z\frac{dX[z]}{dz}$

  Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -n\sum_{n=-\infty}^{\infty} x[n]z^{-n-1} \quad \Rightarrow \quad -z\frac{dX(z)}{dz} = n\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{nx[n]\}$$

Zamanda Kaydırma
İndisin 0 olduğu yerin kaydırılması

$x(n-n_0)$ sinyalinin z-dönüşümü $X(z)z^{-n_0}$ dir.

Örnek: $x(n)=\{1,2,5,7,0,1\}$ ise

$\uparrow$

$a)\ x_1(n)=x(n+2)$

$b)x_2(n)=x(n-2)$

sinyallerinin z-dönüşümünü bulunuz.

a) $X_1(z)=X(z)z^2$

$X(z)=z^2+2z+5+7z^{-1}+z^{-1}$

$X_1(z)=z^4+2z^3+5z^2+7z+z^{-1}$

b) $X_2(z)=X(z)z^{-2}$

$=1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}$

Convolution

Time Domain	u(k)	* (convolution)	v(k)	=	y(k)
	$\mathcal{Z}$ ↓ ↑ $\mathcal{Z}^{-1}$		$\mathcal{Z}$ ↓ ↑ $\mathcal{Z}^{-1}$		$\mathcal{Z}$ ↓ ↑ $\mathcal{Z}^{-1}$
Z Domain	U(z)	. (multiplication)	V(z)	=	Y(z)

Evrişim - Konvalizasyon

Örnek: Aşağıdaki ayrık zamanlı sinyallerin evrişimini bulunuz.

$x_1(n) = \{\underset{\uparrow}{1}, -2, 1\}$ n=0

$x_1(n) * x_2(n) = X_1(z)X_2(z)$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & diğer \end{cases}$$

Çözüm 1: $x_1(n)$'in z-dönüşümü

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$x_2(n)$'in z-dönüşümü

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

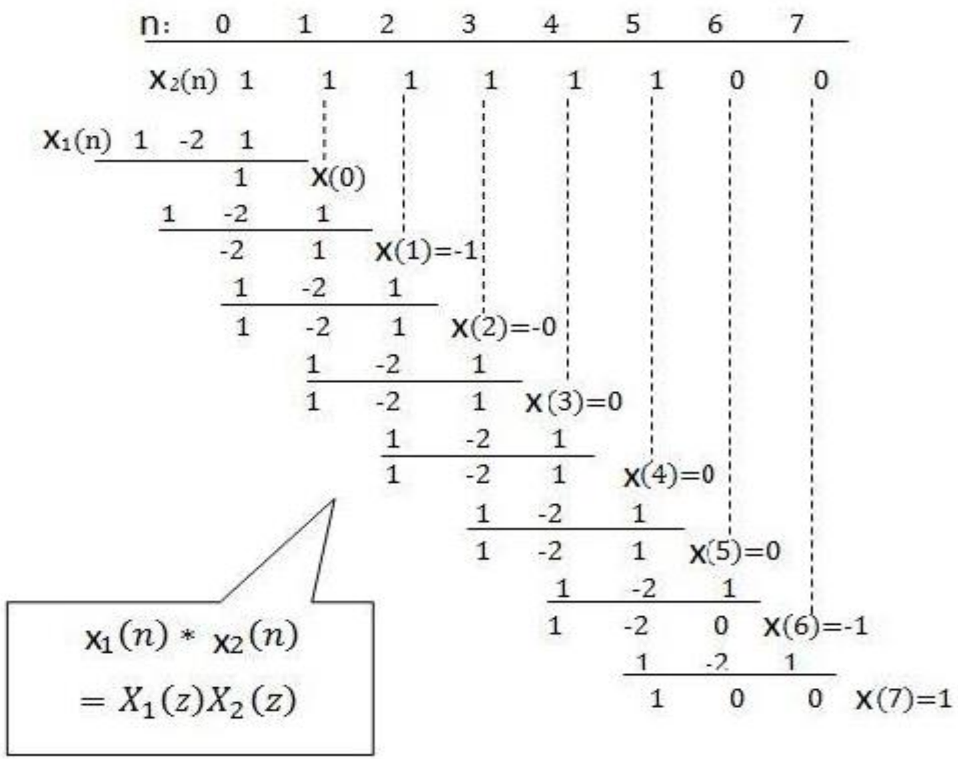
$$X(z) = X_1(z)X_2(z) =$$

$$(1 - 2z^{-1} + z^{-2})(1 - z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$= 1 - z^{-1} + z^{-6} + z^{-7}$$

Ters z-dönüşüm alınırsa, $x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$

Çözüm 2: $x_1(n)$ ile $x_2(n)$ sinyallerinin evrişimi



Region of Convergence (ROC)

Kutuplar ve Sıfırlar

- Bir z-dönüşümünün kutupları, o z-dönüşümünü sonsuza götüren z değerleridir.
- Bir z-dönüşümünün sıfırları, o z-dönüşümünü sıfıra götüren z değerleridir.

Örnek: $X(z) = \frac{1}{1-az^{-1}}$ fonksiyonunun kutup ve sıfırlarını bulunuz.

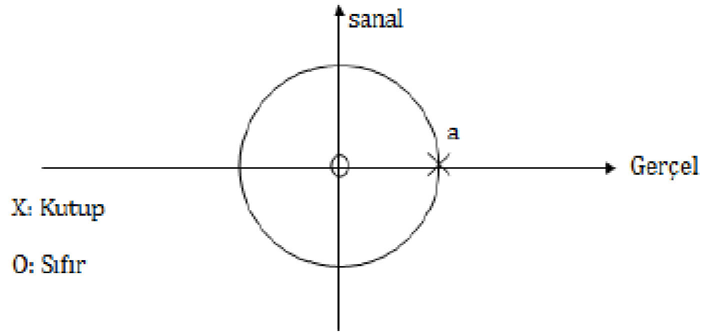
Çözüm: $X(z)$ fonksiyonunun pay ve paydasını z ile çarpalım

$$---> \quad X(z) = \frac{z}{z-a}$$

Sıfır: $z = 0 \quad --\rightarrow \quad X(z) = 0$

Kutup: $z = a \quad --\rightarrow \quad X(z) = \infty$

$X(z)$ fonksiyonunun kutup-sıfır grafiğini çiziniz.

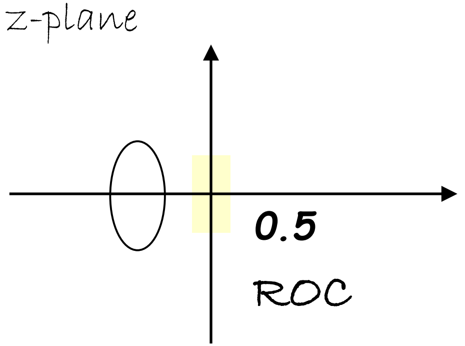


Z-Transform

- Region of Convergence (ROC)
  In general, the z-transform is an infinite sum! This means it (the z-transform) may not exist for all values of z. More specifically, it is the value of r = |z| that is important. If $x(n) = (0.5)^n u(n)$, then

$$X(z) = \sum_{n=0}^{\infty} (0.5)^n z^{-n} = \sum_{n=0}^{\infty} (0.5 z^{-1})^n = \frac{1}{1 - 0.5 z^{-1}}$$

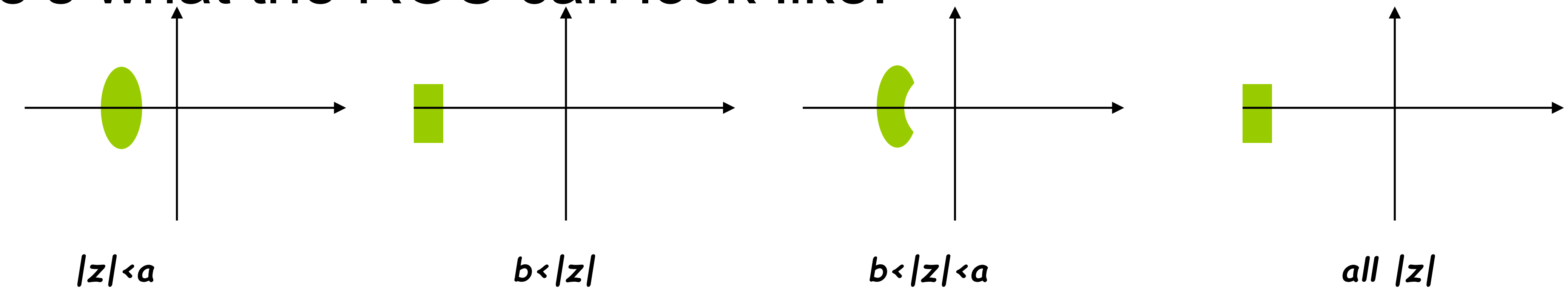
only if |z|>0.5!



Z-Transform

- Region of Convergence

Here's what the ROC can look like:



Z-Transform Yakınsama Bölgesi

Örnek: Aşağıdaki dizilerin z-dönüşümünü hesaplayınız ve yakınsama bölgesini bulunuz.

a) $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

n=0

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = X(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

$X(z)$ $z = 0$ *ve* $z = \infty$ değerleri için tanımsız olmakta, diğer bütün z değerleri için sonlu olmaktadır. O halde yakınsama bölgesi, z=0 ve z=∞ hariç bütün z düzlemidir.

More on ROC

- **Example:** $x[n] = -a^n u[-n-1] \quad \text{(left-sided signal)}$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(-a^n u[-n-1]\right) z^{-n} = -\sum_{n=-\infty}^{-1} \left(a^{-1} z\right)^n$$

$$= 1 - \frac{1-(a^{-1}z)^{\infty}}{1-a^{-1}z} = ?$$

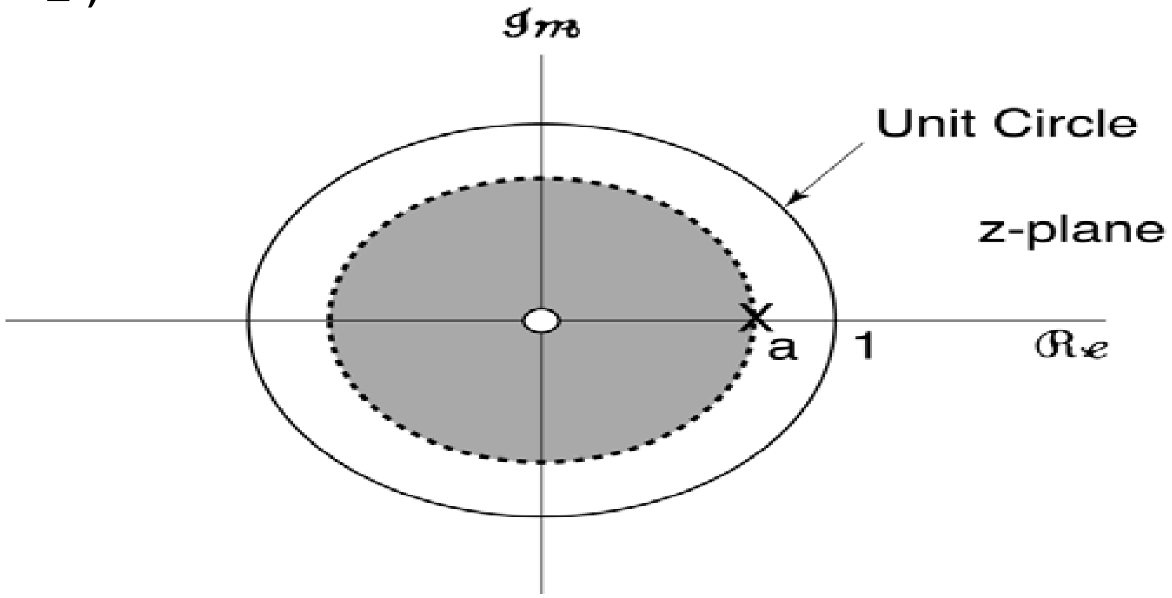
If: $\left|a^{-1}z\right| < 1, \text{ or, } |z| < |a|$

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1-a^{-1}z}{1-a^{-1}z} - \frac{1}{1-a^{-1}z}$$

$$= \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= \frac{1}{1-az^{-1}}$$

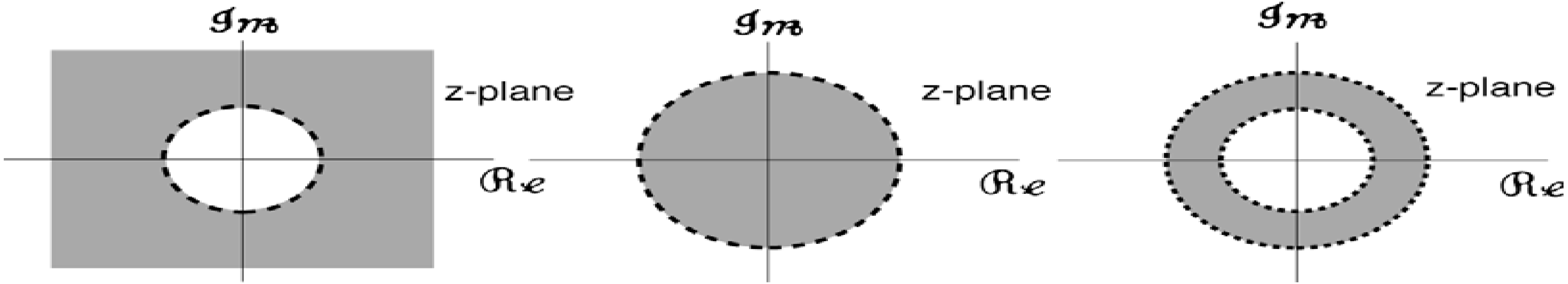
$$= \frac{z}{z-a}$$



The *z*-Transform is the same, but the region of convergence is different.

Properties of the ROC (Cont.)

- If $x[n]$ is a two-sided sequence, and if $|z| = r_0$ is in the ROC, then the ROC consists of a ring in the z-plane including $|z| = r_0$.



right-sided left-sided two-sided

$$x[n] = b^{|n|} \qquad b > 0$$

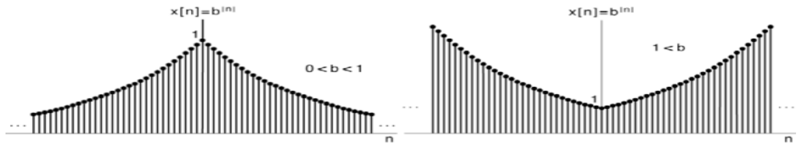
- **Example:**

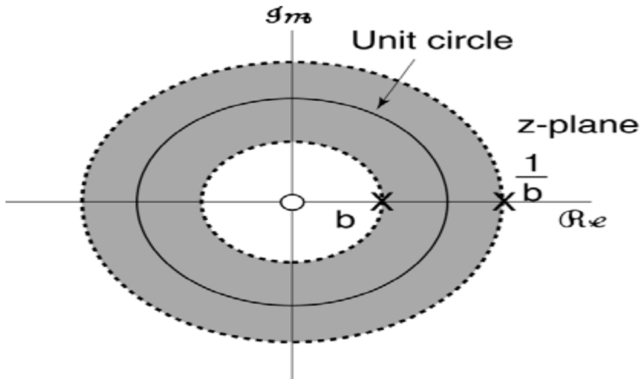
$$x[n] = b^n u[n] + b^{-n} u[-n]$$

$$b^n u[n] \quad \Leftrightarrow \quad \frac{1}{1 - bz^{-1}} \qquad |z| > b$$

$$b^{-n} u[-n-1] \quad \Leftrightarrow \quad \frac{-1}{1 - b^{-1} z^{-1}} \qquad |z| < \frac{1}{b}$$

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1} z^{-1}} \qquad \frac{1}{b} < |z| < b$$





Example: Determine the z-transform of the following signals

(a) $x[n] = [1, 2, 5, 7, 0, 1]$

Solution: $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5},$

ROC: entire z plane except $z = 0$

(b) $y[n] = [1, 2, 5, 7, 0, 1]$ (arrow under 7)

Solution: $Y(z) = z^{2} + 2z + 5 + 7z^{-1} + z^{-3}$

ROC: entire z-plane except $z = 0$ and $z = \infty$.

(c) $z[n] = [0, 0, 1, 2, 5, 7, 0, 1]$

Solution: $z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7},$ **ROC: all z except $z=0$**

Block Diagram Manipulation

Example 1: Running Average Algorithm

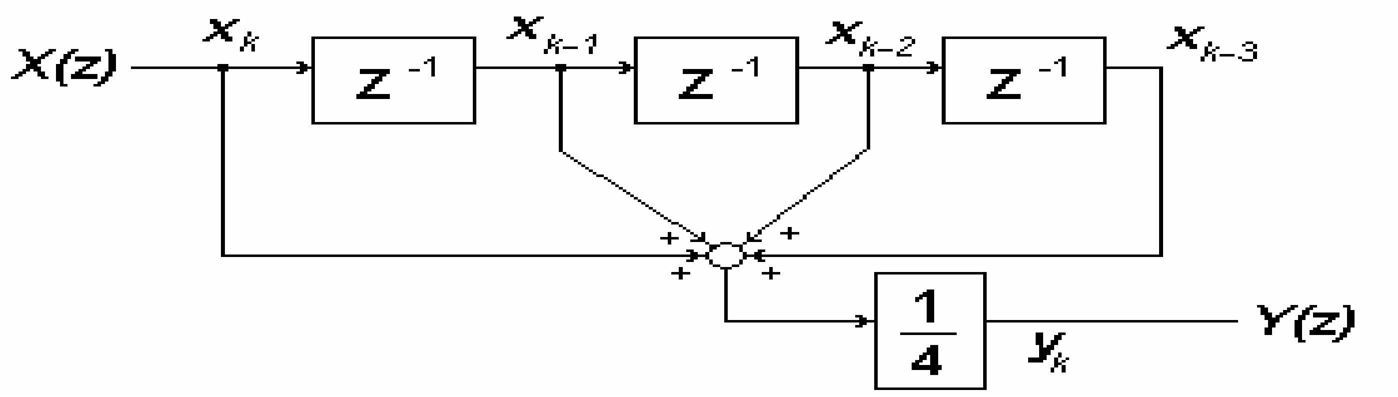
$$y_k = \frac{x_k + x_{k-1} + x_{k-2} + x_{k-3}}{4}$$

(Non-Recursive)

$$Y(z) = X(z)\frac{1 + z^{-1} + z^{-2} + z^{-3}}{4} = X(z)\frac{z^3 + z^2 + z + 1}{4z^4}$$

Z Transform

Block Diagram



Transfer Function

$$\frac{Y}{X}(z) = \frac{z^3 + z^2 + z + 1}{4z^4}$$

Note: Each [Z^{-1}] block can be thought of as a memory cell, storing the previously applied value.

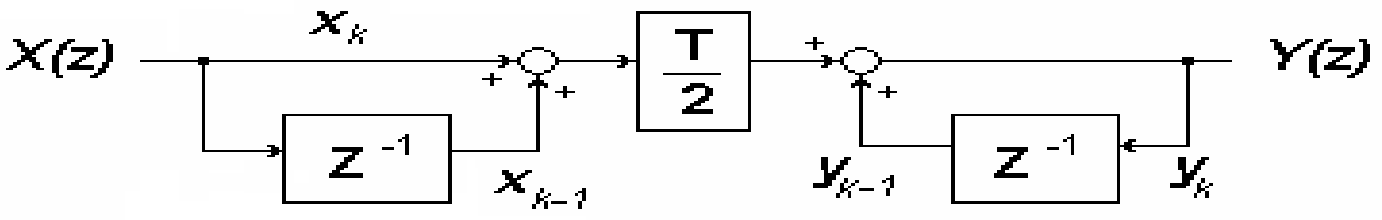
Example 2: Trapezoidal Integrator

$$y_k = y_{k-1} + \left[x_k + x_{k-1}\right]\frac{T}{2}$$ (Recursive)

$$Y(z) = z^{-1}Y(z) + \left[X(z) + z^{-1}X(z)\right]\frac{T}{2}$$ Z Transform

$$Y(z) = X(z)\left[\frac{1+z^{-1}}{1-z^{-1}}\right]\frac{T}{2} = X(z)\left[\frac{z+1}{z-1}\right]\frac{T}{2}$$

Block Diagram

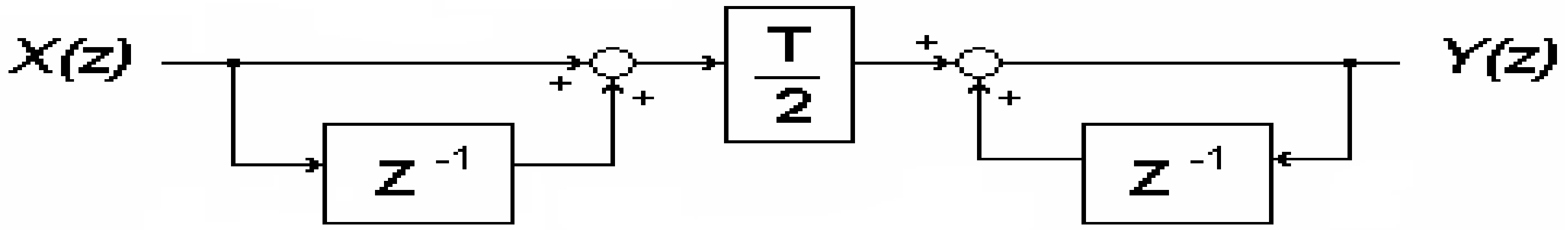


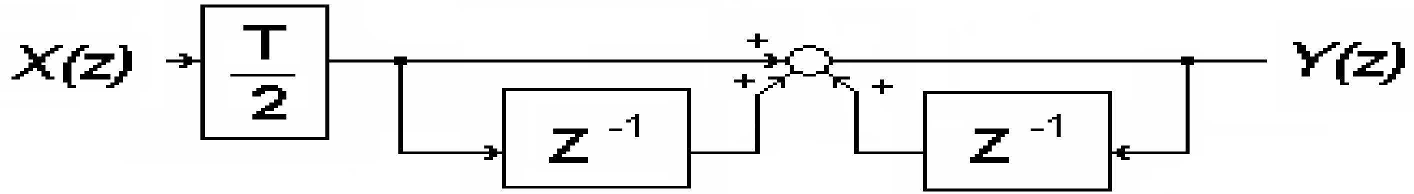
Transfer Function

$$\frac{Y(z)}{X(z)} = \frac{T}{2}\left[\frac{z+1}{z-1}\right]$$

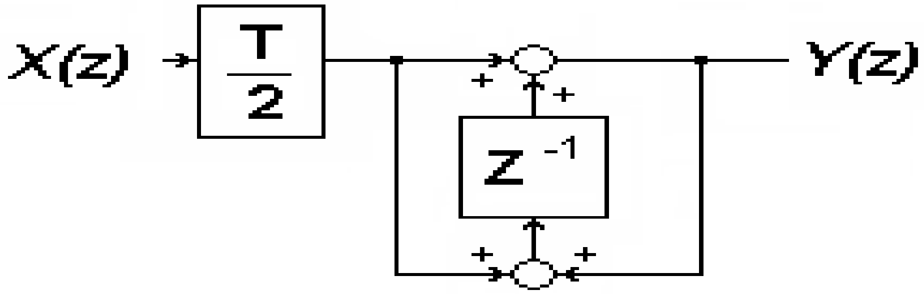
Ex. 2 (cont) Block Diagram Manipulation

Intuitive Structure



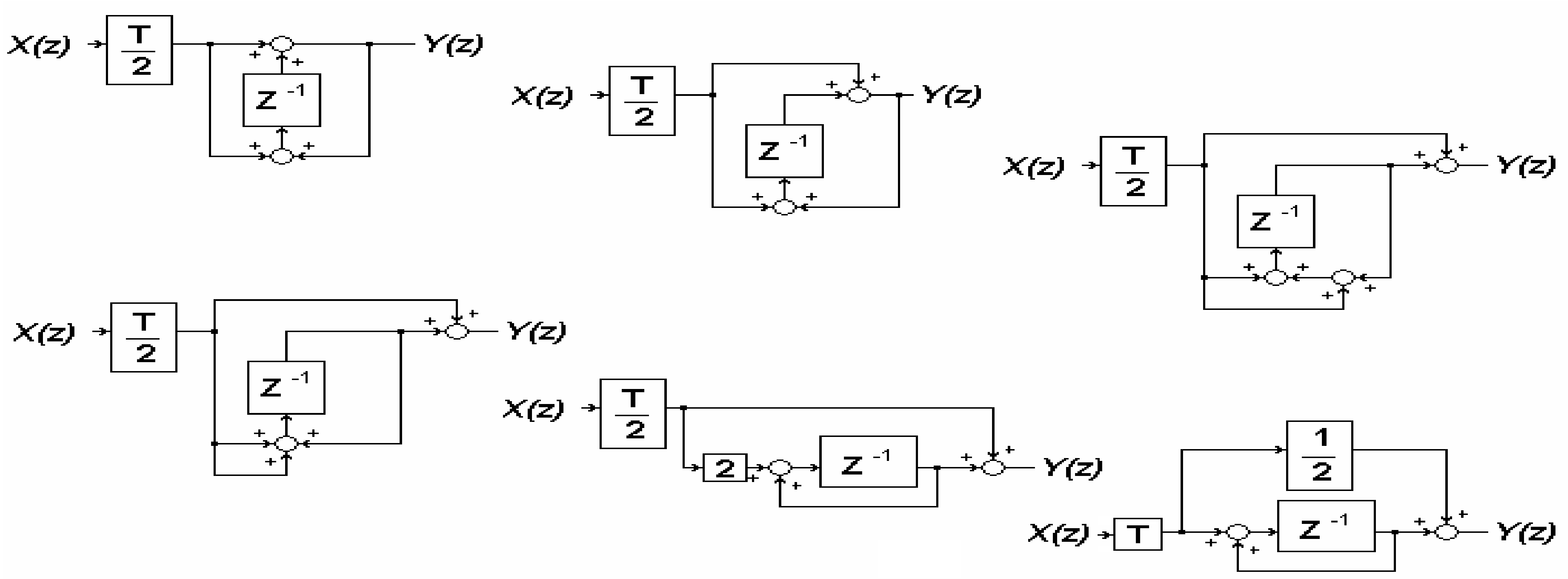


Equivalent Structure



Explicit representation of x_{k-1} and y_{k-1} has been lost, but memory element usage has been reduced from two to one.

Ex. 2 (cont) More Block Diagram Manipulation



$$\frac{Y(z)}{X(z)} = \frac{T}{2}\left[\frac{z+1}{z-1}\right]$$

Note that the final form is equivalent to a rectangular integrator with an additive forward path. In a PI compensator, this path can be absorbed by the proportional term, so there is no advantage to be gained by implementing a trapezoidal integrator.

Inverse Z - Transform

Polinom Bölümü ve Genişletme Metodu

Burada, pay ve payda z^n, z^{n-1}, z^{n-2} veya z^{-1}, z^{-2}, z^{-3} şeklinde yazılarak, uzun bölme işlemi yapılır ve elde edilen bölme sonucunun ters z dönüşümü alınır.

Inverse z Transform Example

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \; 0.5 < |z| < 2$$

Right-sided signals have ROC's that are outside a circle and left-sided signals have ROC's that are inside a circle. Using

$$\alpha^n \, u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| > |\alpha|$$

$$-\alpha^n \, u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| < |\alpha|$$

We get

$$(0.5)^n \, u[n] + (-2)^n \, u[-n-1] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \; 0.5 < |z| < 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n \, \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

We get

$$\left[(0.5)^n - (-2)^n\right]\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n \, \mathrm{u}[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n\right]\mathrm{u}[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

An Example – Complete Solution

$$U(z) = \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} \quad \longrightarrow \quad U(z) = c_0 + \frac{c_1}{z-2} + \frac{c_2}{z-4}$$

$$c_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \frac{3z^2 - 14z + 14}{z^2 - 6z + 8} = 3$$

$$U_2(z) = (z-2)\frac{3z^2 - 14z + 14}{z^2 - 6z + 8} = \frac{3z^2 - 14z + 14}{z-4}$$

$$c_1 = U_2(2) = \frac{3 \cdot 2^2 - 14 \cdot 2 + 14}{2-4} = 1$$

$$U_3(z) = (z-4)\frac{3z^2 - 14z + 14}{z^2 - 6z + 8} = \frac{3z^2 - 14z + 14}{z-2}$$

$$c_2 = U_3(4) = \frac{3 \cdot 4^2 - 14 \cdot 4 + 14}{4-2} = 3$$

$$U(z) = 3 + \frac{1}{z-2} + \frac{3}{z-4}$$

$$u(k) = \begin{cases} 3, & k = 0 \\ 2^{k-1} + 3 \cdot 4^{k-1}, & k > 0 \end{cases}$$

Solving Difference Equations

Solving Difference Equations

The unilateral z transform is well suited to solving difference equations with initial conditions. For example,

$$\mathrm{y}[n+2] - \frac{3}{2}\mathrm{y}[n+1] + \frac{1}{2}\mathrm{y}[n] = (1/4)^n \quad , \quad \text{for } n \geq 0$$

$$\mathrm{y}[0] = 10 \quad \text{and} \quad \mathrm{y}[1] = 4$$

z transforming both sides,

$$z^2\left[\mathrm{Y}(z) - \mathrm{y}[0] - z^{-1}\mathrm{y}[1]\right] - \frac{3}{2}z\left[\mathrm{Y}(z) - \mathrm{y}[0]\right] + \frac{1}{2}\mathrm{Y}(z) = \frac{z}{z - 1/4}$$

the initial conditions are called for systematically.

Solving Difference Equations

Applying initial conditions and solving,

$$\mathrm{Y}(z) = z\left(\frac{16/3}{z-1/4} + \frac{4}{z-1/2} + \frac{2/3}{z-1}\right)$$

and

$$\mathrm{y}[n] = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]\mathrm{u}[n]$$

This solution satisfies the difference equation and the initial conditions.

Transfer Function

z-Transform Properties

An LTI system has a transfer function

$$\mathrm{H}(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{z - 1/2}{z^2 - z + 2/9} \quad , \quad |z| > 2/3$$

Using the time-shifting property of the z transform draw a block diagram realization of the system.

$$\mathrm{Y}(z)\left(z^2 - z + 2/9\right) = \mathrm{X}(z)(z - 1/2)$$

$$z^2\,\mathrm{Y}(z) = z\mathrm{X}(z) - (1/2)\,\mathrm{X}(z) + z\mathrm{Y}(z) - (2/9)\,\mathrm{Y}(z)$$

$$\mathrm{Y}(z) = z^{-1}\,\mathrm{X}(z) - (1/2)\,z^{-2}\,\mathrm{X}(z) + z^{-1}\,\mathrm{Y}(z) - (2/9)\,z^{-2}\,\mathrm{Y}(z)$$

Matlab

Matlab

- *ztrans(f)* finds the Z-Transform of f. By default, the independent variable is n and the transformation variable is z. If f does not contain n, ztrans uses symvar.

Mathematical Operation	**MATLAB Command**
$\lim_{x \to 0} f(x)$	limit(f)
$\lim_{x \to a} f(x)$	limit(f, x, a) or limit(f, a)
$\lim_{x \to a^-} f(x)$	limit(f, x, a, 'left')
$\lim_{x \to a^+} f(x)$	limit(f, x, a, 'right')

Matlab - Örnekler

```
clc
clear all
close all

x = [9 2 3 4 5];
b=0;
n=length(x);
y=sym('z');
for i=1:n
    b=b+x(i)*y^(1-i);
end
display(b);
```

```
b = 2/z + 3/z^2 + 4/z^3 + 5/z^4 + 9
```

```
clc
clear all
close all
syms z k
f=0.5.^k;
a=ztrans(f)
pretty (a)
```

```
a = z/(z - 1/2)

   z
-------
z - 1/2
```

```
clc
clear all
close all
syms z k
F=z/(z-0.5);
a=iztrans(F,k)
pretty(a)
```

```
a = (1/2)^k
      k
 (1/2)
```

Usage Notes

- These slides were gathered from the presentations published on the internet. I would like to thank who prepared slides and documents.
- Also, these slides are made publicly available on the web for anyone to use
- If you choose to use them, I ask that you alert me of any mistakes which were made and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides.

Sincerely,

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